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Research Article

Improvement of Aczél's Inequality and Popoviciu's Inequality

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We generalize and sharpen Aczél's inequality and Popoviciu's inequality by means of two classical inequalities, a unified improvement of Aczél's inequality and Popoviciu's inequality is given. As application, an integral inequality of Aczél-Popoviciu type is established.

1. Introduction

In 1956, Aczél [1] proved the following result:

$$\left(a_1^2-\sum_{i=2}^n a_i^2\right)\left(b_1^2-\sum_{i=2}^n b_i^2\right)\leq\left(a_1b_1-\sum_{i=2}^n a_ib_i\right)^2, \tag{1.1}$$

where a_i, b_i $(i=1,2,\dots,n)$ are positive numbers such that $a_1^2-\sum_{i=2}^n a_i^2>0$ or $b_1^2-\sum_{i=2}^n b_i^2>0$. This inequality is called Aczél's inequality.

It is well known that Aczél's inequality has important applications in the theory of functional equations in non-Euclidean geometry. In recent years, this inequality has attracted the interest of many mathematicians and has motivated a large number of research papers involving different proofs, various generalizations, improvements, and applications (see [2–11] and references therein). We state here a brief history on improvement of Aczél's inequality.

Popoviciu [12] first presented an exponential extension of Aczél's inequality, as follows.

Theorem 1.1. *Let $p>0$, $q>0$, $1/p+1/q=1$, and let a_i, b_i $(i=1,2,\ldots,n)$ be positive numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\le a_1b_1-\sum_{i=2}^n a_ib_i. \tag{1.2}$$

Wu and Debnath [13] generalized inequality (1.2) in the following form.

Theorem 1.2. *Let $p>0$, $q>0$, and let a_i, b_i $(i=1,2,\ldots,n)$ be positive numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\le n^{1-\min\{p^{-1}+q^{-1},1\}}a_1b_1-\sum_{i=2}^n a_ib_i. \tag{1.3}$$

In a recent paper [14], Wu established a sharp and generalized version of Popoviciu's inequality as follows.

Theorem 1.3. *Let $p>0$, $q>0$, $1/p+1/q\ge 1$, and let a_i, b_i $(i=1,2,\ldots,n)$ be positive numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\le a_1b_1-\left(\sum_{i=2}^n a_ib_i\right)-\frac{a_1b_1}{\max\{p,q,1\}}\left(\sum_{i=2}^n\left(\frac{a_i^p}{a_1^p}-\frac{b_i^q}{b_1^q}\right)\right)^2. \tag{1.4}$$

In this paper, we show a new sharp and generalized version of Popoviciu's inequality, which is a unified improvement of Aczél's inequality and Popoviciu's inequality. In Section 4, the obtained result will be used to establish an integral inequality of Aczél-Popoviciu type.

2. Lemmas

In order to prove the theorem in Section 3, we first introduce the following lemmas.

Lemma 2.1 (generalized Hölder inequality [15, page 20]). *Let $a_{ij}>0$, $\lambda_j\ge 0$ $(i=1,2,\ldots,n,\ j=1,2,\ldots,m)$, and let $\lambda_1+\lambda_2+\cdots+\lambda_m=1$. Then*

$$\prod_{j=1}^m\left(\sum_{i=1}^n a_{ij}\right)^{\lambda_j}\ge\sum_{i=1}^n\prod_{j=1}^m a_{ij}^{\lambda_j} \tag{2.1}$$

with equality holding if and only if $a_{11}/a_{1j}=a_{21}/a_{2j}=\cdots=a_{n1}/a_{nj}$ $(j=2,3,\ldots,m)$ for $\lambda_1\lambda_2\cdots\lambda_n\ne 0$.

Lemma 2.2 (mean value inequality [16, page 17]). *Let $x_i>0$, $\lambda_i>0$ $(i=1,2,\ldots,n)$ and let $\lambda_1+\lambda_2+\cdots+\lambda_n=1$. Then*

$$\sum_{i=1}^n\lambda_ix_i\ge\prod_{i=1}^n x_i^{\lambda_i} \tag{2.2}$$

with equality holding if and only if $x_1=x_2=\cdots=x_n$.

Lemma 2.3. *Let $p_1 \geq p_2 \geq \cdots \geq p_m > 0$, $1/p_1 + 1/p_2 + \cdots + 1/p_m = 1$, $0 < x_j < 1$ $(j = 1,2,\ldots,m)$, and let $x_{m+1} = x_1$, $p_{m+1} = p_1$. Then*

$$\prod_{j=1}^{m} x_j + \prod_{j=1}^{m}\left(1 - x_j^{p_j}\right)^{1/p_j} \leq 1 - \frac{1}{2p_1}\sum_{j=1}^{m}\left(x_j^{p_j} - x_{j+1}^{p_{j+1}}\right)^2 \tag{2.3}$$

with equality holding if and only if $x_1^{p_1} = x_2^{p_2} = \cdots = x_m^{p_m}$.

Proof. From hypotheses in Lemma 2.3, it is easy to verify that

$$\begin{gathered}
\frac{1}{p_m} \geq \frac{1}{p_{m-1}} \geq \cdots \geq \frac{1}{p_2} \geq \frac{1}{p_1} > 0, \\
\frac{1}{2p_2} - \frac{1}{2p_1} \geq 0, \frac{1}{2p_3} - \frac{1}{2p_2} \geq 0, \ldots, \frac{1}{2p_m} - \frac{1}{2p_{m-1}} \geq 0, \frac{1}{2p_m} - \frac{1}{2p_1} \geq 0, \\
\begin{aligned}
&\frac{1}{2p_1} + \frac{1}{2p_1} + \left(\frac{1}{2p_2} - \frac{1}{2p_1}\right) + \frac{1}{2p_2} + \frac{1}{2p_2} + \left(\frac{1}{2p_3} - \frac{1}{2p_2}\right) + \cdots + \frac{1}{2p_{m-2}} + \frac{1}{2p_{m-2}} \\
&\quad + \left(\frac{1}{2p_{m-1}} - \frac{1}{2p_{m-2}}\right) + \frac{1}{2p_{m-1}} + \frac{1}{2p_{m-1}} + \left(\frac{1}{2p_m} - \frac{1}{2p_{m-1}}\right) + \frac{1}{2p_1} + \frac{1}{2p_1} + \left(\frac{1}{2p_m} - \frac{1}{2p_1}\right) \\
&\quad = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1.
\end{aligned}
\end{gathered} \tag{2.4}$$

Hence, by using Lemma 2.1 we obtain

$$\begin{aligned}
&\left[x_1^{p_1} + \left(1 - x_2^{p_2}\right)\right]^{1/2p_1}\left[x_2^{p_2} + \left(1 - x_1^{p_1}\right)\right]^{1/2p_1}\left[x_2^{p_2} + \left(1 - x_2^{p_2}\right)\right]^{1/2p_2 - 1/2p_1} \\
&\quad \times \left[x_2^{p_2} + \left(1 - x_3^{p_3}\right)\right]^{1/2p_2}\left[x_3^{p_3} + \left(1 - x_2^{p_2}\right)\right]^{1/2p_2}\left[x_3^{p_3} + \left(1 - x_3^{p_3}\right)\right]^{1/2p_3 - 1/2p_2} \\
&\qquad\qquad\qquad\qquad \vdots \\
&\quad \times \left[x_{m-2}^{p_{m-2}} + \left(1 - x_{m-1}^{p_{m-1}}\right)\right]^{1/2p_{m-2}} \\
&\quad \times \left[x_{m-1}^{p_{m-1}} + \left(1 - x_{m-2}^{p_{m-2}}\right)\right]^{1/2p_{m-2}}\left[x_{m-1}^{p_{m-1}} + \left(1 - x_{m-1}^{p_{m-1}}\right)\right]^{1/2p_{m-1} - 1/2p_{m-2}} \\
&\quad \times \left[x_{m-1}^{p_{m-1}} + \left(1 - x_m^{p_m}\right)\right]^{1/2p_{m-1}}\left[x_m^{p_m} + \left(1 - x_{m-1}^{p_{m-1}}\right)\right]^{1/2p_{m-1}}\left[x_m^{p_m} + \left(1 - x_m^{p_m}\right)\right]^{1/2p_m - 1/2p_{m-1}} \\
&\quad \times \left[x_m^{p_m} + \left(1 - x_1^{p_1}\right)\right]^{1/2p_1}\left[x_1^{p_1} + \left(1 - x_m^{p_m}\right)\right]^{1/2p_1}\left[x_m^{p_m} + \left(1 - x_m^{p_m}\right)\right]^{1/2p_m - 1/2p_1} \\
&\geq x_1^{p_1/2p_1} x_2^{p_2/2p_1} x_2^{p_2/2p_2 - p_2/2p_1} x_2^{p_2/2p_2} \cdots x_{m-1}^{p_{m-1}/2p_{m-2}} x_{m-1}^{p_{m-1}/2p_{m-1} - p_{m-1}/2p_{m-2}} x_{m-1}^{p_{m-1}/2p_{m-1}} \\
&\quad \times x_m^{p_m/2p_{m-1}} x_m^{p_m/2p_m - p_m/2p_{m-1}} x_m^{p_m/2p_1} x_m^{p_m/2p_m - p_m/2p_1} x_1^{p_1/2p_1} \\
&\quad + \left(1 - x_1^{p_1}\right)^{1/2p_1}\left(1 - x_2^{p_2}\right)^{1/2p_1}\left(1 - x_2^{p_2}\right)^{1/2p_2 - 1/2p_1}\left(1 - x_2^{p_2}\right)^{1/2p_2} \\
&\quad \cdots \left(1 - x_{m-1}^{p_{m-1}}\right)^{1/2p_{m-2}}\left(1 - x_{m-1}^{p_{m-1}}\right)^{1/2p_{m-1} - 1/2p_{m-2}}\left(1 - x_{m-1}^{p_{m-1}}\right)^{1/2p_{m-1}} \\
&\quad \times \left(1 - x_m^{p_m}\right)^{1/2p_{m-1}}\left(1 - x_m^{p_m}\right)^{1/2p_m - 1/2p_{m-1}}\left(1 - x_m^{p_m}\right)^{1/2p_1}\left(1 - x_m^{p_m}\right)^{1/2p_m - 1/2p_1} \\
&\quad \times \left(1 - x_1^{p_1}\right)^{1/2p_1},
\end{aligned} \tag{2.5}$$

which is equivalent to

$$\begin{aligned}
&\left[1-\left(x_1^{p_1}-x_2^{p_2}\right)^2\right]^{1/2p_1}\left[1-\left(x_2^{p_2}-x_3^{p_3}\right)^2\right]^{1/2p_2} \\
&\qquad \cdots\left[1-\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2\right]^{1/2p_{m-1}}\left[1-\left(x_m^{p_m}-x_1^{p_1}\right)^2\right]^{1/2p_1} \\
&\qquad \geq x_1x_2\cdots x_m+\left(1-x_1^{p_1}\right)^{1/p_1}\left(1-x_2^{p_2}\right)^{1/p_2}\cdots\left(1-x_m^{p_m}\right)^{1/p_m}.
\end{aligned} \tag{2.6}$$

On the other hand, it follows from Lemma 2.2 that

$$\begin{aligned}
&\frac{1}{2p_1}\left[1-\left(x_1^{p_1}-x_2^{p_2}\right)^2\right]+\frac{1}{2p_2}\left[1-\left(x_2^{p_2}-x_3^{p_3}\right)^2\right]+\cdots+\frac{1}{2p_{m-1}}\left[1-\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2\right] \\
&\qquad +\frac{1}{2p_1}\left[1-\left(x_m^{p_m}-x_1^{p_1}\right)^2\right]+\left(\frac{1}{2p_2}+\frac{1}{2p_3}+\cdots+\frac{1}{2p_{m-1}}+\frac{1}{p_m}\right)\cdot 1 \\
&\quad \geq \left[1-\left(x_1^{p_1}-x_2^{p_2}\right)^2\right]^{1/2p_1}\left[1-\left(x_2^{p_2}-x_3^{p_3}\right)^2\right]^{1/2p_2} \\
&\qquad \cdots\left[1-\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2\right]^{1/2p_{m-1}}\left[1-\left(x_m^{p_m}-x_1^{p_1}\right)^2\right]^{1/2p_1},
\end{aligned} \tag{2.7}$$

this yields

$$\begin{aligned}
&\left[1-\left(x_1^{p_1}-x_2^{p_2}\right)^2\right]^{1/2p_1}\left[1-\left(x_2^{p_2}-x_3^{p_3}\right)^2\right]^{1/2p_2}\cdots\left[1-\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2\right]^{1/2p_{m-1}}\left[1-\left(x_m^{p_m}-x_1^{p_1}\right)^2\right]^{1/2p_1} \\
&\quad \leq \left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}\right)-\frac{1}{2p_1}\left(x_1^{p_1}-x_2^{p_2}\right)^2-\frac{1}{2p_2}\left(x_2^{p_2}-x_3^{p_3}\right)^2 \\
&\qquad -\cdots-\frac{1}{2p_{m-1}}\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2-\frac{1}{2p_1}\left(x_m^{p_m}-x_1^{p_1}\right)^2 \\
&\quad \leq 1-\frac{1}{2p_1}\left[\left(x_1^{p_1}-x_2^{p_2}\right)^2+\left(x_2^{p_2}-x_3^{p_3}\right)^2+\cdots+\left(x_{m-1}^{p_{m-1}}+x_m^{p_m}\right)^2+\left(x_m^{p_m}-x_1^{p_1}\right)^2\right].
\end{aligned} \tag{2.8}$$

Combining inequalities (2.6) and (2.8) leads to inequality (2.3). In addition, from Lemmas 2.1 and 2.2, we can easily deduce that the equality holds in both (2.6) and (2.8) if and only if $x_1^{p_1}=x_2^{p_2}=\cdots=x_m^{p_m}$, and thus we obtain the condition of equality in (2.3). The proof of Lemma 2.3 is complete. □

3. Improvement of Aczél's inequality and Popoviciu's inequality

Theorem 3.1. *Let $p_1\geq p_2\geq\cdots\geq p_m>0$, $1/p_1+1/p_2+\cdots+1/p_m=1$, $a_{ij}>0$, $a_{1j}^{p_j}-\sum_{i=2}^{n}a_{ij}^{p_j}>0$ $(i=1,2,\ldots,n,\ j=1,2,\ldots,m)$, and let $p_{m+1}=p_1$, $a_{im+1}=a_{i1}$ $(i=1,2,\ldots,n)$. Then one has the following inequality:*

$$\prod_{j=1}^{m}\left(a_{1j}^{p_j}-\sum_{i=2}^{n}a_{ij}^{p_j}\right)^{1/p_j}\leq\prod_{j=1}^{m}a_{1j}-\sum_{i=2}^{n}\prod_{j=1}^{m}a_{ij}-\frac{a_{11}a_{12}\cdots a_{1m}}{2p_1}\sum_{j=1}^{m}\left(\sum_{i=2}^{n}\left(\frac{a_{ij}^{p_j}}{a_{1j}^{p_j}}-\frac{a_{ij+1}^{p_{j+1}}}{a_{1j+1}^{p_{j+1}}}\right)\right)^2. \tag{3.1}$$

Equality holds in (3.1) if and only if $a_{11}^{p_1}/a_{1j}^{p_j}=a_{21}^{p_1}/a_{2j}^{p_j}=\cdots=a_{n1}^{p_1}/a_{nj}^{p_j}$ $(j=2,3,\ldots,m)$.

Proof. Since by hypotheses in Theorem 3.1 we have

$$0 < \frac{\left(a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}\right)^{1/p_j}}{\left(a_{1j}^{p_j}\right)^{1/p_j}} < 1 \quad (j = 1,2,\ldots,m), \tag{3.2}$$

it follows from Lemma 2.3, with a substitution $x_j = (a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j})^{1/p_j}/(a_{1j}^{p_j})^{1/p_j}$ $(j = 1,2,\ldots,m)$ in (2.3), that

$$\begin{aligned}
&\prod_{j=1}^{m}\left(\frac{a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}}{a_{1j}^{p_j}}\right)^{1/p_j} + \prod_{j=1}^{m}\left(\frac{\sum_{i=2}^{n} a_{ij}^{p_j}}{a_{1j}^{p_j}}\right)^{1/p_j} \\
&\quad \leq 1 - \frac{1}{2p_1}\sum_{j=1}^{m}\left(\frac{a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}}{a_{1j}^{p_j}} - \frac{a_{1j+1}^{p_{j+1}} - \sum_{i=2}^{n} a_{ij+1}^{p_{j+1}}}{a_{1j+1}^{p_{j+1}}}\right)^2,
\end{aligned} \tag{3.3}$$

which is equivalent to

$$\prod_{j=1}^{m}\left(a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}\right)^{1/p_j} \leq \prod_{j=1}^{m} a_{1j} - \prod_{j=1}^{m}\left(\sum_{i=2}^{n} a_{ij}^{p_j}\right)^{1/p_j} - \frac{a_{11}a_{12}\cdots a_{1m}}{2p_1}\sum_{j=1}^{m}\left(\sum_{i=2}^{n}\left(\frac{a_{ij}^{p_j}}{a_{1j}^{p_j}} - \frac{a_{ij+1}^{p_{j+1}}}{a_{1j+1}^{p_{j+1}}}\right)\right)^2, \tag{3.4}$$

where equality holds if and only if $(\sum_{i=2}^{n} a_{ij}^{p_j})/a_{1j}^{p_j} = (\sum_{i=2}^{n} a_{ij+1}^{p_{j+1}})/a_{1j+1}^{p_{j+1}}$ $(j = 1,2,\ldots,m)$, that is, if and only if $a_{11}^{p_1}/a_{1j}^{p_j} = (\sum_{i=2}^{n} a_{i1}^{p_1})/(\sum_{i=2}^{n} a_{ij}^{p_j})$ $(j = 2,3,\ldots,m)$.

On the other hand, using Lemma 2.1 gives

$$\prod_{j=1}^{m}\left(\sum_{i=2}^{n} a_{ij}^{p_j}\right)^{1/p_j} \geq \sum_{i=2}^{n}\prod_{j=1}^{m} a_{ij}, \tag{3.5}$$

where equality holds if and only if $a_{21}^{p_1}/a_{2j}^{p_j} = a_{31}^{p_1}/a_{3j}^{p_j} = \cdots = a_{n1}^{p_1}/a_{nj}^{p_j}$ $(j = 2,3,\ldots,m)$.

Combining inequalities (3.4) and (3.5) leads to the desired inequality (3.1). By means of the conditions of equality in (3.4) and (3.5), it is easy to conclude that there is equality in (3.1) if and only if $a_{11}^{p_1}/a_{1j}^{p_j} = a_{21}^{p_1}/a_{2j}^{p_j} = \cdots = a_{n1}^{p_1}/a_{nj}^{p_j}$ $(j = 2,3,\ldots,m)$. This completes the proof of Theorem 3.1. □

As a consequence of Theorem 3.1, putting $m = 2$, $p_1 = p$, $p_2 = q$, $a_{i1} = a_i$, $a_{i2} = b_i$ $(i = 1,2,\ldots,n)$ in (3.1), we get the following.

Corollary 3.2. *Let $p \geq q > 0$, $1/p + 1/q = 1$, and let a_i, b_i $(i = 1,2,\ldots,n)$ be positive numbers such that $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_1^q - \sum_{i=2}^{n} b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p}\left(b_1^q - \sum_{i=2}^{n} b_i^q\right)^{1/q} \leq a_1b_1 - \left(\sum_{i=2}^{n} a_ib_i\right) - \frac{a_1b_1}{p}\left(\sum_{i=2}^{n}\left(\frac{a_i^p}{a_1^p} - \frac{b_i^q}{b_1^q}\right)\right)^2 \tag{3.6}$$

with equality holding if and only if $a_1^p/b_1^q = a_2^p/b_2^q = \cdots = a_n^p/b_n^q$.

A simple application of Corollary 3.2 yields the following sharp version of Popoviciu's inequality.

Corollary 3.3. *Let* $p>0$, $q>0$, $1/p+1/q=1$, *and let* a_i, b_i $(i=1,2,\dots,n)$ *be positive numbers such that* $a_1^p-\sum_{i=2}^n a_i^p>0$ *and* $b_1^q-\sum_{i=2}^n b_i^q>0$. *Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\le a_1b_1-\left(\sum_{i=2}^n a_ib_i\right)-\frac{a_1b_1}{\max\{p,q\}}\left(\sum_{i=2}^n\left(\frac{a_i^p}{a_1^p}-\frac{b_i^q}{b_1^q}\right)\right)^2, \tag{3.7}$$

with equality holding if and only if $a_1^p/b_1^q=a_2^p/b_2^q=\cdots=a_n^p/b_n^q$.

Obviously, inequalities (3.1), (3.6), and (3.7) are the improvement of Aczél's inequality and Popoviciu's inequality.

4. Integral version of Aczél-Popoviciu-type inequality

As application of Theorem 3.1, we establish here an interesting integral inequality of Aczél-Popoviciu type.

Theorem 4.1. *Let* $p_1\ge p_2\ge\cdots\ge p_m>0$, $1/p_1+1/p_2+\cdots+1/p_m=1$, $B_j>0$ $(j=1,2,\dots,m)$, *let* f_j *be positive Riemann integrable functions on* $[a,b]$ *such that* $B_j^{p_j}-\int_a^b f_j^{p_j}(x)dx>0$ *for all* $j=1,2,\dots,m$, *and let* $B_{m+1}=B_1$, $p_{m+1}=p_1$, $f_{m+1}=f_1$. *Then one has the following inequality:*

$$\begin{aligned}&\prod_{j=1}^m\left(B_j^{p_j}-\int_a^b f_j^{p_j}(x)dx\right)^{1/p_j}\\&\quad\le\prod_{j=1}^m B_j-\int_a^b\left(\prod_{j=1}^m f_j(x)\right)dx-\frac{B_1B_2\cdots B_m}{2p_1}\sum_{j=1}^m\left(\int_a^b\left(\frac{f_j^{p_j}(x)}{B_j^{p_j}}-\frac{f_{j+1}^{p_{j+1}}(x)}{B_{j+1}^{p_{j+1}}}\right)dx\right)^2.\end{aligned} \tag{4.1}$$

Proof. For any positive integer n, we choose an equidistant partition of $[a,b]$ as

$$\begin{gathered}a<a+\frac{b-a}{n}<\cdots<a+\frac{b-a}{n}i<\cdots<a+\frac{b-a}{n}(n-1)<b,\\\Delta x_i=\frac{b-a}{n},\quad i=1,2,\dots,n.\end{gathered} \tag{4.2}$$

Since the hypothesis $B_j^{p_j}-\int_a^b f_j^{p_j}(x)dx>0$ $(j=1,2,\dots,m)$ implies that

$$B_j^{p_j}-\lim_{n\to\infty}\sum_{i=1}^n f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}>0\quad(j=1,2,\dots,m), \tag{4.3}$$

there exists a positive integer N such that

$$B_j^{p_j}-\sum_{i=1}^n f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}>0\quad\forall n>N,\ j=1,2,\dots,m. \tag{4.4}$$

Applying Theorem 3.1, one obtains for any $n > N$ the following inequality:

$$\begin{aligned}
&\prod_{j=1}^{m}\left[B_j^{p_j}-\sum_{i=1}^{n} f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right]^{1/p_j} \\
&\quad\le \prod_{j=1}^{m} B_j-\sum_{i=1}^{n}\left(\prod_{j=1}^{m} f_j\left(a+\frac{i(b-a)}{n}\right)\right)\left(\frac{b-a}{n}\right)^{1/p_1+1/p_2+\cdots+1/p_m} \\
&\qquad-\frac{B_1B_2\cdots B_m}{2p_1}\sum_{j=1}^{m}\left[\sum_{i=1}^{n}\left(\frac{1}{B_j^{p_j}} f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right.\right. \\
&\qquad\qquad\left.\left.-\frac{1}{B_{j+1}^{p_{j+1}}} f_{j+1}^{p_{j+1}}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right)\right]^2.
\end{aligned} \tag{4.5}$$

Note that $1/p_1+1/p_2+\cdots+1/p_m = 1$, the above inequality can be transformed to

$$\begin{aligned}
&\prod_{j=1}^{m}\left[B_j^{p_j}-\sum_{i=1}^{n} f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right]^{1/p_j} \\
&\quad\le \prod_{j=1}^{m} B_j-\sum_{i=1}^{n}\left(\prod_{j=1}^{m} f_j\left(a+\frac{i(b-a)}{n}\right)\right)\left(\frac{b-a}{n}\right) \\
&\qquad-\frac{B_1B_2\cdots B_m}{2p_1}\sum_{j=1}^{m}\left[\sum_{i=1}^{n}\left(\frac{1}{B_j^{p_j}} f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\right.\right. \\
&\qquad\qquad\left.\left.-\frac{1}{B_{j+1}^{p_{j+1}}} f_{j+1}^{p_{j+1}}\left(a+\frac{i(b-a)}{n}\right)\right)\frac{b-a}{n}\right]^2,
\end{aligned} \tag{4.6}$$

where equality holds if and only if $f_j^{p_j}(a+i(b-a)/n)/B_j^{p_j} = f_{j+1}^{p_{j+1}}(a+i(b-a)/n)/B_{j+1}^{p_{j+1}}$ for all $i = 1,2,\ldots,n$ $(j = 1,2,\ldots,m)$.

In view of the hypotheses that f_j are positive Riemann integrable functions on $[a,b]$ and $p_j > 0$ $(j = 1,2,\ldots,m)$, we conclude that $\prod_{j=1}^{m} f_j$ and $f_j^{p_j}$ $(j = 1,2,\ldots,m)$ are also integrable on $[a,b]$. Passing the limit as $n \to \infty$ in both sides of inequality (4.6), we obtain the inequality (4.1). The proof of Theorem 4.1 is complete. □

Remark 4.2. Motivated by the proof of Theorem 4.1, we propose here a conjecture.

Conjecture 4.3. Suppose that $p_1 \ge p_2 \ge \cdots \ge p_m > 0$, $1/p_1+1/p_2+\cdots+1/p_m = 1$, $B_j > 0$ $(j = 1,2,\ldots,m)$, suppose also that $f_j \in L^{p_j}[a,b]$, $B_j^{p_j} - \int_a^b |f_j(x)|^{p_j}dx > 0$ for all $j = 1,2,\ldots,m$, let $B_{m+1} = B_1$, $p_{m+1} = p_1$, $f_{m+1} = f_1$. Then the following inequality holds true:

$$\begin{aligned}
&\prod_{j=1}^{m}\left(B_j^{p_j}-\int_a^b |f_j(x)|^{p_j}dx\right)^{1/p_j} \\
&\quad\le\prod_{j=1}^{m} B_j-\int_a^b\left(\prod_{j=1}^{m}|f_j(x)|\right)dx-\frac{B_1B_2\cdots B_m}{2p_1}\sum_{j=1}^{m}\left(\int_a^b\left(\frac{|f_j(x)|^{p_j}}{B_j^{p_j}}-\frac{|f_{j+1}(x)|^{p_{j+1}}}{B_{j+1}^{p_{j+1}}}\right)dx\right)^2
\end{aligned} \tag{4.7}$$

with equality holding if and only if $|f_j(x)|^{p_j}/B_j^{p_j} = |f_{j+1}(x)|^{p_{j+1}}/B_{j+1}^{p_{j+1}}$ $(j = 1,2,\ldots,m)$ almost everywhere on $[a,b]$.

As a consequence of Theorem 4.1, putting $m = 2$, $p_1 = p$, $p_2 = q$, $B_1 = A$, $B_2 = B$, $f_1 = f$, $f_2 = g$ in (4.1), we obtain the following.

Corollary 4.4. *Let $p \geq q > 0$, $1/p + 1/q = 1$, $A > 0$, $B > 0$, and let f, g be positive Riemann integrable functions on $[a,b]$ such that $A^p - \int_a^b f^p(x)dx > 0$ and $B^q - \int_a^b g^q(x)dx > 0$. Then*

$$\begin{aligned}&\left(A^p - \int_a^b f^p(x)dx\right)^{1/p}\left(B^q - \int_a^b g^q(x)dx\right)^{1/q}\\&\quad\leq AB - \int_a^b f(x)g(x)dx - \frac{AB}{p}\left(\int_a^b\left(\frac{f^p(x)}{A^p} - \frac{g^q(x)}{B^q}\right)dx\right)^2.\end{aligned}\tag{4.8}$$

Further, from Corollary 4.4 we have the following.

Corollary 4.5. *Let $p > 0$, $q > 0$, $1/p + 1/q = 1$, $A > 0$, $B > 0$, and let f, g be positive Riemann integrable functions on $[a,b]$ such that $A^p - \int_a^b f^p(x)dx > 0$ and $B^q - \int_a^b g^q(x)dx > 0$. Then*

$$\begin{aligned}&\left(A^p - \int_a^b f^p(x)dx\right)^{1/p}\left(B^q - \int_a^b g^q(x)dx\right)^{1/q}\\&\quad\leq AB - \int_a^b f(x)g(x)dx - \frac{AB}{\max\{p,q\}}\left(\int_a^b\left(\frac{f^p(x)}{A^p} - \frac{g^q(x)}{B^q}\right)dx\right)^2.\end{aligned}\tag{4.9}$$

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References

[1] J. Aczél, "Some general methods in the theory of functional equations in one variable. New applications of functional equations," *Uspekhi Matematicheskikh Nauk (N.S.)*, vol. 11, no. 3(69), pp. 3–68, 1956 (Russian).

[2] Y. J. Cho, M. Matić, and J. Pečarić, "Improvements of some inequalities of Aczél's type," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 1, pp. 226–240, 2001.

[3] X.-H. Sun, "Aczél-Chebyshev type inequality for positive linear functions," *Journal of Mathematical Analysis and Applications*, vol. 245, no. 2, pp. 393–403, 2000.

[4] L. Losonczi and Z. Páles, "Inequalities for indefinite forms," *Journal of Mathematical Analysis and Applications*, vol. 205, no. 1, pp. 148–156, 1997.

[5] A. M. Mercer, "Extensions of Popoviciu's inequality using a general method," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, Article 11, pp. 4 pages, 2003.

[6] V. Mascioni, "A note on Aczél type inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 5, Article 69, pp. 5 pages, 2002.

[7] S. S. Dragomir and B. Mond, "Some inequalities of Aczél type for Gramians in inner product spaces," *Nonlinear Functional Analysis and Applications*, vol. 6, no. 3, pp. 411–424, 2001.

[8] R. Bellman, "On an inequality concerning an indefinite form," *The American Mathematical Monthly*, vol. 63, no. 2, pp. 108–109, 1956.

[9] P. M. Vasić and J. E. Pečarić, "On the Jensen inequality for monotone functions," *Analele Universității din Timişoara. Seria Matematică-Informatică*, vol. 17, no. 1, pp. 95–104, 1979.
[10] J. C. Kuang, *Applied Inequalities*, Hunan Education Press, Changsha, China, 2nd edition, 1993.
[11] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, vol. 61, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
[12] T. Popoviciu, "On an inequality," *Gazeta Matematica si Fizica. Seria A*, vol. 11 (64), pp. 451–461, 1959 (Romanian).
[13] S. Wu and L. Debnath, "Generalizations of Aczél's inequality and Popoviciu's inequality," *Indian Journal of Pure and Applied Mathematics*, vol. 36, no. 2, pp. 49–62, 2005.
[14] S. Wu, "A further generalization of Aczél's inequality and Popoviciu's inequality," *Mathematical Inequalities and Application*, vol. 10, no. 3, 2007.
[15] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, New York, NY, USA, 1983.
[16] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2nd edition, 1952.

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